An Asymptotic Study of Discretized Transport Equations in the Fokker-Planck Limit

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Abstract

Recent analyses have shown that the Fokker-Planck (FP) equation is an asymptotic limit of the transport equation given a forward-peaked scattering kernel satisfying certain constraints. In this paper we study discretized transport equations in the same limit, both by asymptotic analysis and by numerical testing. We show that spatially discretized discrete-ordinates transport solutions can be accurate in this limit if and only if the scattering operator is handled in a certain non-standard way.

1 Introduction

The Fokker-Planck (FP) equation is a classical approximation to linear transport theory in optically thick systems with highly anisotropic scattering. Recent analyses (Pomraning, 1992; Börgers, 1994) have shown that the FP equation is an asymptotic limit of the transport equation; that is, that as scattering cross sections become large and sufficiently forward-peaked, the transport solution approaches the solution of the Fokker-Planck equation. In this paper we analyze discretized transport equations in the same limit. We show that the leading-order solution to the discrete-ordinates (S_N) transport equation satisfies a pseudospectral discretization of the FP equation, provided that the scattering term is handled in a certain way, which we describe, and that the analytic transport solution limits to an analytic FP solution. We also show that the leading-order solutions to several common one- and two-dimensional spatial discretizations of the S_N equations satisfy reasonable discretizations of the FP equations, given the same provisions. We also provide numerical results; these invariably agree with theoretical predictions. This work provides a theoretical foundation for the application of S_N methods to certain problems with forward-peaked scattering.

2 Analytic Transport

We briefly review Pomraning's asymptotic analysis (Pomraning, 1992) of the fully analytic transport equation in the Fokker-Planck limit, which we restrict to the monoenergetic case. The analytic transport equation in general geometry is:

$$\begin{aligned}&\mathbf{\Omega}\cdot\boldsymbol{\nabla}\psi\left(\mathbf{r},\ \mathbf{\Omega}\right)+\left(\sigma_a\left(\mathbf{r}\right)+\sigma_{s0}\left(\mathbf{r}\right)\right)\psi\left(\mathbf{r},\mathbf{\Omega}\right)\\ =\ &\sum_{n=0}^{\infty}\sum_{m=m_L}^{m_H}\left(\frac{2n+1}{c_{norm}}\right)Y_{nm}\left(\mathbf{\Omega}\right)\varphi_{nm}\left(\mathbf{r}\right)\sigma_{sn}\left(\mathbf{r}\right)+q\left(\mathbf{r},\mathbf{\Omega}\right),\end{aligned}\tag{1}$$

where

$$\sigma_{sn}=\int_{-1}^{1}d\mu_0 P_n\left(\mu_0\right)\sigma_s\left(\mu_0\right),\qquad \varphi_{nm}\left(\mathbf{r}\right)=\int_{4\pi}d\mathbf{\Omega}'Y_{nm}^{*}\left(\mathbf{\Omega}'\right)\psi\left(\mathbf{r},\mathbf{\Omega}'\right),\tag{2}$$

and where $\sigma_s\left(\mu_0\right)$ is the cross section for scattering through an angle whose cosine is μ_0. The parameter c_{norm} is a normalization factor that depends on n, m, and the normalization chosen for the definition of $Y_{nm}\left(\mathbf{\Omega}\right)$, and the parameters m_L and m_H depend on the geometry. Following Pomraning, we make the following definitions and scalings:

$$\delta = 1-\langle\overline{\mu_0}\rangle=\frac{\langle\sigma_{s0}\rangle-\langle\sigma_{s1}\rangle}{\langle\sigma_{s0}\rangle},\tag{3a}$$

$$\gamma = \left\langle\overline{\left(1-\mu_0\right)^2}\right\rangle=\frac{4\langle\sigma_{s0}\rangle-6\langle\sigma_{s1}\rangle+2\langle\sigma_{s2}\rangle}{3\langle\sigma_{s0}\rangle},\tag{3b}$$

$$y = \frac{1-\mu_0}{\delta},\tag{3c}$$

$$\sigma_{sn}\left(\mathbf{r}\right) = \frac{\hat{\sigma}_{sn}\left(\mathbf{r}\right)}{\delta},\qquad \sigma_a\left(\mathbf{r}\right)=\hat{\sigma}_a\left(\mathbf{r}\right),\tag{3d}$$

$$\sigma_{tr}\left(\mathbf{r}\right) = \sigma_a\left(\mathbf{r}\right)+\sigma_{s0}\left(\mathbf{r}\right)\left(1-\overline{\mu_0}\left(\mathbf{r}\right)\right),\tag{3e}$$

where $\overline{\mu_0}$ is the average scattering cosine, $\hat{\sigma}_{sn}$ and $\hat{\sigma}_a$ are $O\left(1\right)$, δ and γ are small, and $\langle\cdot\rangle$ indicates a typical value of a position-dependent variable. The idea is to examine what happens to the solution as δ approaches zero. Physically, this corresponds to a diminishing distance between scatters but also a diminishing average scattering angle. These are balanced such that σ_{tr} is $O\left(1\right)$ and independent of δ.

By rewriting the integral for σ_{sn} in terms of y, expanding the integrand in a Taylor series about $\mu_0=1$, and truncating after the second term, Pomraning obtained the asymptotic form for σ_{sn}:

$$\sigma_{sn}\left(\mathbf{r}\right)=\sigma_{s0}\left(\mathbf{r}\right)\left[1-\frac{n\left(n+1\right)}{2}\delta+O\left(\gamma\right)\right].\tag{4}$$

The scattering kernel above is the Fokker-Planck kernel with an extra $O\left(\gamma\right)$ term. Substitution of this expression into the transport equation leads (after some manipulation) to the following intermediate result:

$$
\begin{aligned}
&\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\psi\left(\mathbf{r},\boldsymbol{\Omega}\right)+\hat{\sigma}_a\left(\mathbf{r}\right)\psi\left(\mathbf{r},\boldsymbol{\Omega}\right)\\
&+\frac{\hat{\sigma}_{s0}\left(\mathbf{r}\right)}{\delta}\left\{\psi\left(\mathbf{r},\boldsymbol{\Omega}\right)-\sum_{n=0}^{\infty}\sum_{m=m_L}^{m_H}\left(\frac{2n+1}{c_{norm}}\right)Y_{nm}\left(\boldsymbol{\Omega}\right)\varphi_{nm}\left(\mathbf{r}\right)\right\}\\
=\quad&-\frac{\left(\sigma_{tr}\left(\mathbf{r}\right)-\hat{\sigma}_a\left(\mathbf{r}\right)\right)}{2}\sum_{n=0}^{\infty}\sum_{m=m_L}^{m_H}\left(\frac{2n+1}{c_{norm}}\right)n\left(n+1\right)Y_{nm}\left(\boldsymbol{\Omega}\right)\varphi_{nm}\left(\mathbf{r}\right)\\
&+q\left(\mathbf{r},\boldsymbol{\Omega}\right)+O\left(\frac{\gamma}{\delta}\right). \qquad (5)
\end{aligned}
$$

Equation (5) is an equivalent form for the transport equation that is obtained when the scattering cross section is asymptotically made forward-peaked. When the preceding analysis was reported by Pomraning, he obtained the Fokker-Planck equation by noting that the term in braces in Eq. (5) is identically zero. In order to illuminate the discrete analyses in subsequent sections, however, we wish to proceed by a slightly more formal route. Our goal is to discern how the transport solution, $\psi\left(\mathbf{r},\boldsymbol{\Omega}\right)$, behaves in the limit as δ tends to zero. Therefore we propose the asymptotic ansatz:

$$\psi = \psi^{(0)}+\delta\psi^{(1)}+\delta^2\psi^{(2)}+\cdots, \qquad (6a)$$

$$\varphi_{nm} = \varphi_{nm}^{(0)}+\delta\varphi_{nm}^{(1)}+\delta^2\varphi_{nm}^{(2)}+\cdots, \qquad (6b)$$

where we will be primarily interested in the leading-order term. We insert this ansatz into Eqs. (2) and (5); after some manipulation we obtain the following requirement:

$$\left(I-MD\right)\psi^{(0)}=0, \qquad (7)$$

where M and D are the moments-to-discrete and discrete-to-moments operators, respectively:

$$\left(M\varphi\right)\left(\boldsymbol{\Omega}\right)=\sum_{n=0}^{\infty}\sum_{m=m_L}^{m_H}\left(\frac{2n+1}{c_{norm}}\right)Y_{nm}\left(\boldsymbol{\Omega}\right)\varphi_{nm}, \qquad \left(D\psi\right)_{nm}=\int_{4\pi}d\boldsymbol{\Omega}'Y_{nm}^{*}\left(\boldsymbol{\Omega}'\right)\psi\left(\boldsymbol{\Omega}'\right). \qquad (8)$$

Since $M=D^{-1}$ for analytic transport this condition is satisfied in the current analysis; the net effect then is to eliminate the term in braces in Eq. (5). Although this result is rather obvious in the problem we are now analyzing, we will see later that the discrete version of Eq. (7) is not satisfied for some discretizations.

By substituting the identity:

$$\left[\frac{\partial}{\partial\mu}\left(1-\mu^2\right)\frac{\partial}{\partial\mu}+\left(\frac{1}{1-\mu^2}\right)\frac{\partial^2}{\partial\phi^2}+n\left(n+1\right)\right]Y_{nm}\left(\boldsymbol{\Omega}\right)=0 \qquad (9)$$

into Eq. (5) we obtain a Fokker-Planck equation with an extra $O\left(\gamma/\delta\right)$ term:

$$
\begin{aligned}
&\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\psi^{(0)}\left(\mathbf{r},\boldsymbol{\Omega}\right)+\hat{\sigma}_a\left(\mathbf{r}\right)\psi^{(0)}\left(\mathbf{r},\boldsymbol{\Omega}\right)\\
=\quad&\frac{\sigma_{tr}\left(\mathbf{r}\right)-\hat{\sigma}_a\left(\mathbf{r}\right)}{2}\left[\frac{\partial}{\partial\mu}\left(1-\mu^2\right)\frac{\partial}{\partial\mu}+\left(\frac{1}{1-\mu^2}\right)\frac{\partial^2}{\partial\phi^2}\right]\psi^{(0)}\left(\mathbf{r},\boldsymbol{\Omega}\right)\\
&+q\left(\mathbf{r},\boldsymbol{\Omega}\right)+O\left(\frac{\gamma}{\delta}\right). \qquad (10)
\end{aligned}
$$

The $O(\gamma/\delta)$ term in Eq. (10) is significant. It is a function only of the scattering kernel. The FP equation is not an asymptotic limit of the transport equation unless the scattering kernel is such that $\gamma \to 0$ more rapidly than $\delta \to 0$, i.e., such that

$$\left\langle \overline{(1-\mu_0)^2} \right\rangle \Big/ \langle (1-\bar{\mu}_0) \rangle \to 0 \tag{11}$$

as $\bar{\mu}_0 \to 1$. The Henyey-Greenstein kernel (Henyey, 1941), for example, does not have this limit. For more discussion see (Pomraning, 1992) and (Börgers, 1994).

3 Spatially Analytic Discrete Ordinates

We now turn our attention to the discrete-ordinates discretization of the transport equation. The standard discrete-ordinates version of the transport equation (1) is (with the asymptotic form of σ_{sn} inserted):

$$\begin{aligned} &\mathbf{\Omega}_k \cdot \nabla \psi_k(\mathbf{r}) + (\sigma_a(\mathbf{r}) + \sigma_{s0}(\mathbf{r}))\,\psi_k(\mathbf{r}) \\ = &\sum_{n=0}^{N-1} \sum_{m=m_L}^{m_H} \left(\frac{2n+1}{c_{norm}}\right) Y_{nm}(\mathbf{\Omega}_k)\,\tilde{\varphi}_{nm}(\mathbf{r}) \frac{\hat{\sigma}_{s0}(\mathbf{r})}{\delta} \left[1 - \frac{n(n+1)}{2}\delta + O(\gamma)\right] \\ &+ q(\mathbf{r}, \mathbf{\Omega}_k), \end{aligned} \tag{12}$$

where

$$\tilde{\varphi}_{nm}(\mathbf{r}) \equiv \sum_{k=1}^{K} w_k Y_{nm}^{*}(\mathbf{\Omega}_k)\,\psi_k(\mathbf{r}). \tag{13}$$

Here the w_k and $\mathbf{\Omega}_k$ are the quadrature weights and directions, respectively, of a quadrature of order N. In level-symmetric quadratures $K = N$ in $1D$, $K = N(N+2)/2$ in $2D$, and $K = N(N+2)$ in $3D$. Note that the scattering order in Eq. (12) is truncated at $N-1$. We insert the asymptotic ansatz of Eqs. (6) into Eqs. (12) and (13); we obtain a discrete version of Eq. (7):

$$(I - M_N D_N)\,\psi^{(0)} = 0, \tag{14}$$

A sufficient (although not strictly necessary) condition for satisfying Eq. (14) is that $M_N D_N = I$. This will always be true if M_N and D_N are inverses of each other (as in analytic transport), in which case there are as many moments in the scattering expansion as there are discrete angles. It may or may not be true if there are more moments than discrete angles. It cannot be true if there are fewer moments than angles. (These assertions follow directly from the dimensions of the matrices.) In one-dimensional slab and spherical geometry it will be true if and only if the quadrature set exactly integrates polynomials of degree $2N-2$, as is the case with the Gauss-Legendre (GL) set. In standard multidimensional implementations there are generally more discrete angles than scattering moments, so in these cases $M_N D_N \neq I$. If $M_N D_N \neq I$, then Eq. (14) will be satisfied only if some other (generally non-physical) constraints are met; if Eq.(14) is not satisfied then the asymptotic ansatz of Eqs. (6) is not valid. In such a case there is no $O(1)$ solution to Eqs. (12) and (13).

If we assume that $M_N D_N = I$, the rest of the analysis and Eq. (9) yield:

$$\begin{aligned} & \mathbf{\Omega}_k \cdot \nabla \psi_k^{(0)}(\mathbf{r}) + \hat{\sigma}_a(\mathbf{r})\, \psi_k^{(0)}(\mathbf{r}) \\ = \; & \frac{\sigma_{tr}(\mathbf{r}) - \hat{\sigma}_a(\mathbf{r})}{2} \left\{ \left[\frac{\partial}{\partial \mu}\left(1-\mu^2\right) \frac{\partial}{\partial \mu} + \left(\frac{1}{1-\mu^2} \right) \frac{\partial^2}{\partial \phi^2} \right] \tilde{\psi}^{(0)}(\mathbf{r}, \mathbf{\Omega}) \right\}_{\mathbf{\Omega}=\mathbf{\Omega}_k} \\ & + q(\mathbf{r}, \mathbf{\Omega}_k) + O\left(\frac{\gamma}{\delta}\right), \end{aligned} \tag{15}$$

where in one-dimensional slab and spherical geometry we define $\tilde{\psi}^{(0)}(\mathbf{r}, \mathbf{\Omega})$ to be the $(N-1)$-order polynomial interpolant through the points $\left\{ \mathbf{\Omega}_k, \psi^{(0)}(\mathbf{r}, \mathbf{\Omega}_k) \right\}$. (The definition in multidimensional geometry will be described below). Thus, assuming that $O(\gamma/\delta) \rightarrow 0$ as $\delta \rightarrow 0$, Eq. (15) is a "pseudospectral" discretization (Gottlieb, 1984) of the angular variable in the exact FP equation. (Pseudospectral methods use collocation to determine coefficients in a global function expansion.)

The above discussion indicates that the transformation from discrete values to angular moments and back to discrete values should be the identity. If Eqs. (12) and (13) define the discrete-to-moments and moments-to-discrete transformations, then we will not have the identity unless the quadrature set is Gauss-Legendre in one-dimensional slab or spherical geometry. Given a different quadrature set and/or multidimensional geometry, then, the S_N method may not limit to a discretization of the FP equation unless Eq. (12) and/or (13) is replaced.

Morel (Morel, 1989) reached the same conclusion via a completely different analysis, and offered suggestions for replacing the offending equation(s). The simplest suggestion in one-dimensional slab and spherical geometry is to use for $\tilde{\varphi}_{n0}$ the *exact* moments of the $(N-1)$-order polynomial, $\tilde{\psi}$, that goes through the points $\{\mathbf{\Omega}_k, \psi(\mathbf{r}, \mathbf{\Omega}_k)\}$; i.e. to solve Eq. (2) exactly instead of using Eq. (13), thereby redefining D_N to be M_N^{-1}. Morel labeled this "Galerkin" quadrature, since he derived it by means of a Galerkin weighting method. The use of the exact moments causes Eq. (14) to be satisfied regardless of quadrature set, and Eq. (15) then follows.

In multidimensional geometries the Galerkin quadrature has a more complex definition. Recall that there are fewer moments than discrete angles in standard multi-dimensional implementations of the discrete-ordinates method. For example, level symmetric quadrature sets of order N have $N(N+2)/2$ and $N(N+2)$ quadrature points in two and three dimensions, respectively, whereas there are $N(N+1)/2$ and N^2 spherical harmonics of order $N-1$ or less in the respective dimensions (Lewis, 1993). In order to satisfy Eq. (14) in all circumstances we must first increase the number of spherical harmonics in our flux expansion by using harmonics of higher orders. Morel (Morel, 1989) and Reed (Reed, 1972) proposed suitable spherical harmonic interpolation spaces for multidimensional geometries. For two-dimensional geometries the following interpolation space is suggested:

$$\left\{ Y_{nm} : \begin{array}{cc} 0 \leq m \leq n, & if\ 0 \leq n \leq N-1 \\ 0 < m\ odd \leq N, & if\ n = N \end{array} \right\}. \tag{16}$$

The interpolation space suggested for three dimensions is:

$$\left\{ Y_{nm} : \begin{array}{cc} -n \leq m \leq n, & if\ 0 \leq n \leq N-1, \\ \left(\begin{array}{c} -n \leq m < 0 \\ and\ 0 < m\ odd \leq N \end{array} \right), & if\ n = N \\ -(N+1) \leq m\ even < 0, & if\ n = N+1 \end{array} \right\}. \tag{17}$$

The Galerkin quadrature is then defined by adjusting the limits of the summations in Eq. (12) in order to augment M_N and then redefining $D_N \equiv M_N^{-1}$. As in the one-dimensional case Eq. (14) will be satisfied regardless of the discrete angle set when the Galerkin treatment is used, and Eq. (15) then follows, where $\tilde{\psi}^{(0)}(\mathbf{r}, \mathbf{\Omega})$ is now defined as the spherical harmonic interpolant (corresponding to the selected interpolation space) through the points $\left\{ \mathbf{\Omega}_k, \tilde{\psi}^{(0)}(\mathbf{r}, \mathbf{\Omega}_k) \right\}$.

We remind ourselves that the condition $D_N = M_N^{-1}$ is certainly sufficient for obtaining the correct FP limit, but it is not strictly necessary. We need only to satisfy Eq. (14), i.e. that $\tilde{\psi}$ be in the null space of $I - M_N D_N$. If $M_N D_N \neq I$, then certain angular eigenmodes cannot be present in a stable solution, i.e. $\tilde{\psi}$ must be in a restricted subspace of the domain of $M_N D_N$. It is entirely possible that a clever selection of boundary conditions and sources could result in a solution that does not contain any of the unstable modes; however, this would suggest that physically realistic boundary conditions would give rise to unstable modes. Alternatively, one could filter out the unstable mode components of the scattering source; this would stabilize the solution, but this will yield a different solution than that obtained when exact integrations are used. Our recommendation is to avoid these complications altogether by simply using the exact inverse of M_N.

4 Spatially Discretized Discrete Ordinates

We extend our asymptotic analysis now to discretizations of both angle and space. We will study the diamond difference (DD), the linear discontinuous (LD) and the linear moments (LM) methods (Alcouffe, 1979) as examples of spatial discretizations of the transport equation in one-dimensional slab geometry. In two-dimensional Cartesian geometry we will examine several related finite element methods on rectangles: the bilinear discontinuous (BLD), the lumped bilinear discontinuous (LBLD) and the simple corner balance (SCB) methods (Adams, 1991; Adams, 1997).

The slab geometry DD-S_N discretization of Eq. (1) is given in (Alcouffe, 1979). We substitute in the asymptotic cross section of Eq. (4) and perform our asymptotic analysis. We obtain Eq. (14) as a necessary condition for a leading-order solution. If we assume that $M_N D_N = I$ and that γ has no $O(\delta)$ components, we obtain

$$\begin{aligned} &\mu_k \left[\frac{\psi_{k,i+\frac{1}{2}}^{(0)} - \psi_{k,i-\frac{1}{2}}^{(0)}}{\Delta x_i} \right] + \hat{\sigma}_{ai} \psi_{ki}^{(0)} \\ &= \frac{(\sigma_{tr,i} - \hat{\sigma}_{ai})}{2} \left[\frac{\partial}{\partial \mu} \left(1 - \mu^2\right) \frac{\partial}{\partial \mu} \tilde{\psi}_i^{(0)} \right]_{\mu = \mu_k} + q_i(\mu_k), \end{aligned} \tag{18a}$$

$$\psi_{k,i+\frac{1}{2}}^{(0)} = \begin{cases} \psi_{k,inc}\left(x_{\frac{1}{2}}\right), & \mu_k > 0, \quad i = 0 \\ \psi_{k,inc}\left(x_{I+\frac{1}{2}}\right), & \mu_k < 0, \quad i = I \\ 2\psi_{ki}^{(0)} - \psi_{k,i-\frac{1}{2}}^{(0)}, & \mu_k > 0, \quad i > 0 \\ 2\psi_{k,i+1}^{(0)} - \psi_{k,i+\frac{3}{2}}^{(0)}, & \mu_k < 0, \quad i < I \end{cases} . \tag{18b}$$

Thus, given the above assumptions, the leading-order DD-S_N solution satisfies a DD-pseudospectral discretization of the FP equation. (It is no surprise that these assumptions are required, since they are required even without spatial discretization.)

The slab geometry LD-S_N discretization of Eq. (1) is given in (Alcouffe, 1979). Our analysis again obtains Eq. (14) as a necessary condition. Given the usual assumptions, we obtain Eq. (18a) and the following:

$$\begin{aligned} & 3\mu_k \left(\frac{\psi_{k,i+\frac{1}{2}}^{(0)} + \psi_{k,i-\frac{1}{2}}^{(0)} - 2\psi_{ki}^{(0)}}{\Delta x_i} \right) + \hat{\sigma}_{ai}\psi_{ki}^{x(0)} \\ = & \frac{(\sigma_{tr,i} - \hat{\sigma}_{ai})}{2} \left[\frac{\partial}{\partial\mu}\left(1-\mu^2\right)\frac{\partial}{\partial\mu}\tilde{\psi}_i^{x(0)} \right]_{\mu=\mu_k} + q_i^x(\mu_k), \end{aligned} \tag{19a}$$

$$\psi_{k,i+\frac{1}{2}}^{(0)} = \begin{cases} \psi_{k,inc}\left(x_{\frac{1}{2}}\right), & \mu_k > 0, \quad i = 0 \\ \psi_{k,inc}\left(x_{I+\frac{1}{2}}\right), & \mu_k < 0, \quad i = I \\ \psi_{ki}^{(0)} + \psi_{ki}^{x(0)}, & \mu_k > 0, \quad i \geq 1 \\ \psi_{k,i+1}^{(0)} - \psi_{k,i+1}^{x(0)}, & \mu_k < 0, \quad i < I \end{cases} , \tag{19b}$$

where first spatial moments are defined:

$$f_i^x \equiv \frac{3}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} dx \left[\frac{x - x_i}{\Delta x_i/2} \right] f(x) . \tag{20}$$

Therefore the leading-order LD-S_N solution satisfies an LD-pseudospectral discretization of the FP equation, given the previously stated constraints on the cross section and the quadrature.

The slab geometry LM-S_N discretization of Eq. (1) is given in (Alcouffe, 1979). We again obtain Eq. (14) by our analysis; the usual assumptions yield Eqs. (18a) and (19), i.e. the leading-order **LM**-S_N solution is identical to the leading-order **LD**-S_N solution: it satisfies an LD-pseudospectral discretization of the FP equation. Once again, the constraints on the cross section and the quadrature apply.

We have also studied some related discontinuous finite element (FEM) schemes in two-dimensional Cartesian geometry: the bilinear discontinuous (BLD), the lumped bilinear discontinuous (LBLD) and the simple corner balance (SCB) methods. These discretizations are given in (Adams, 1991; Adams, 1997). We restrict our analysis to rectangular grids, in which case all of the above discretizations have the same form, differing only in the elements of certain matrices. Our asymptotic analysis yields Eq. (14) as a necessary condition. The usual assumptions yield the result that the leading-order FEM-S_N solution satisfies a FEM-pseudospectral discretization of the FP equation.

5 Numerical Results

Now we present numerical results that support our Fokker-Planck asymptotic analyses. The specific analytic problem we will examine in slab geometry is defined by Eqs. (1), (2), and

$$\psi(0,\mu) = \delta(1-\mu),\ 0<\mu\le 1, \tag{21a}$$

$$\psi(L,\mu) = 0,\ -1\le\mu<0, \tag{21b}$$

$$\sigma_s(\mu_0) = C(\delta)\exp\left(-\frac{1-\mu_0}{\delta}\right), \qquad \sigma_a = 0, \tag{21c}$$

where the value of $C(\delta)$ is such that $\sigma_{tr} = 0.1$. This problem was examined in (Börgers, 1996). For this cross section $\gamma \to 2\delta^2$ as $\delta \to 0$, so the $O(\gamma/\delta)$ term in Eq. (10) vanishes in the FP limit. Thus the problem above is asymptotically described by the FP equation as $\delta \to 0$.

We examine S_N solutions to this problem near the Fokker-Planck limit. For these studies we use $L = 20$ and $\delta = 0.001$. As boundary conditions we use a quasi-Mark approximation to Eqs. (21a) and (21b) in which all incoming fluxes are set to zero except at the quadrature direction $\mu_{\max}$ closest to $\mu = 1$; we set $\psi(0,\mu_{\max}) = w_{\max}^{-1}$, where $w_{\max}$ is the corresponding quadrature weight from the original quadrature set (not from the Galerkin quadrature defined by the M_N^{-1} matrix). This boundary condition has the effect of preserving the contribution to the boundary scalar flux from the beam in Eq. (21a). Figure 1 shows the scalar flux throughout the slab as calculated by the LM-S_8 method using both the Lobatto-Galerkin and Gaussian quadratures. For these calculations we have used a very fine spatial mesh (200 cells, $\sigma_{tr}\Delta x = 0.01$) to minimize the effects of spatial discretization so that we may concentrate on the effects of the angular differencing. These S_N results compare favorably with a pseudospectral Fokker-Planck solution of order 32, especially the Lobatto results. We remark that a stable solution could not be obtained with the Lobatto set without the Galerkin quadrature treatment. These results demonstrate the need to use discrete-to-moments and moments-to-discrete operators that are inverses of each other, as well as the fact that S_N solutions limit to Fokker-Planck solutions under proper conditions.

We now examine the effects of spatial discretization on the S_N solutions. In Figure 2 we show LM-S_8 solutions as $\delta \to 0$ for a mesh with only 10 cells. We also plot the corresponding LD-pseudospectral Fokker-Planck solution as well as the highly refined FP solution that we showed in Figure 1. This figure shows that as $\delta \to 0$, the spatially discrete S_8 solution approaches the spatially discrete FP solution that we predicted in the previous section, and that this solution is an excellent approximation of the analytic solution.

Although we do not plot the results here, we have also performed similar calculations in *x-y* geometry. As $\delta \to 0$, the DFEM-S_N solution approaches the DFEM-FP solution of the same order, as predicted.

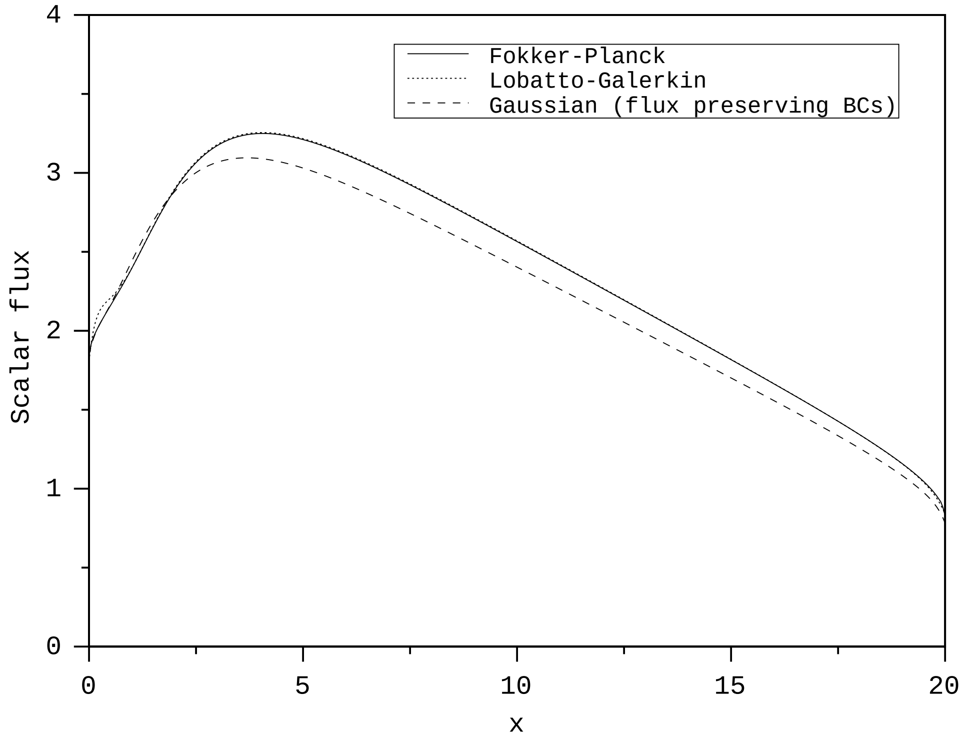

Figure 1: S_8 scalar fluxes, $\delta = 0.001$. (Lobatto-Galerkin and F-P are nearly coincident.)

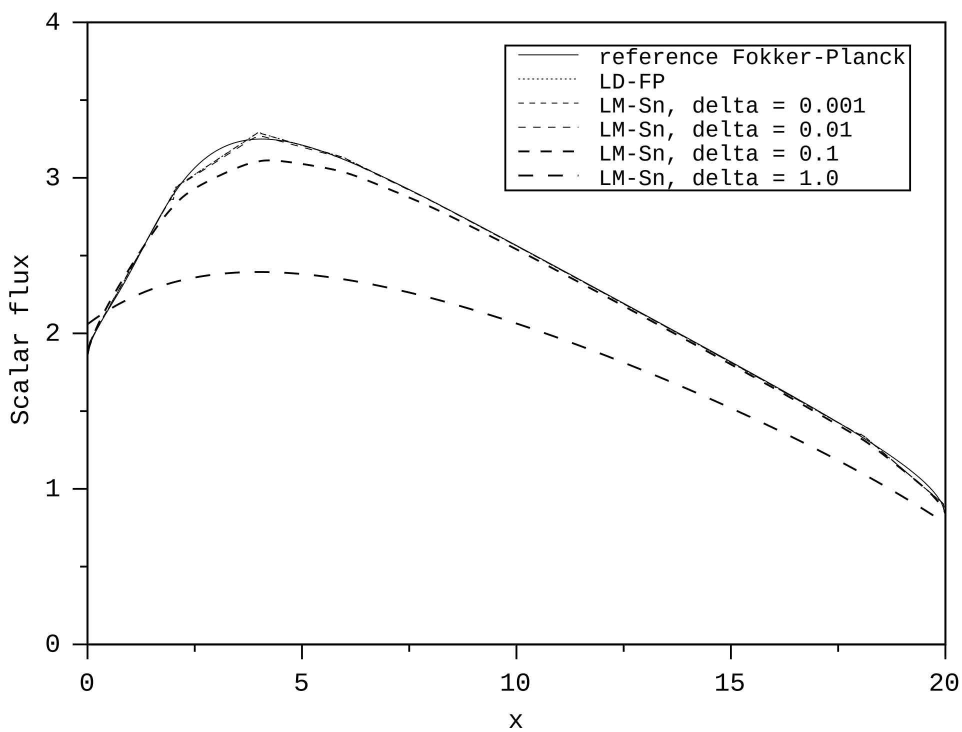

Figure 2: LM-S_8 scalar fluxes. (The LD-FP and LM-S_8-0.001 curves are nearly coincident.)

6 Conclusions

In summary, previous studies have shown that some transport problems with strongly forward-peaked scattering are described by the Fokker-Planck equation, while others are not; this is a function strictly of the scattering kernel. Our analyses of the spatially analytic discrete ordinates (S_N) equations reveal that if the analytic solution does satisfy a FP equation and if $M_N D_N = I$, where M_N is the moments-to-discrete operator of the S_N method and D_N is the discrete-to-moments operator, then the S_N solution will satisfy a pseudospectral discretization of the Fokker-Planck equation. If $M_N D_N \neq I$, then the S_N method will fail in the FP limit unless additional constraints are satisfied. Since $M_N D_N \neq I$ in most standard S_N implementations, these methods will fail to produce reasonable results for forward-peaked scattering problems unless their scattering treatments are modified. We also studied spatially discretized S_N equations in the same limit. We found that if the spatially analytic S_N solution satisfies a pseudospectral FP equation, then the solutions to several spatially discrete S_N schemes will satisfy spatially discretized pseudospectral FP equations. These results provide some theoretical basis for the use of S_N methods for problems with strongly forward-peaked scattering.

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